Integral equations for three-dimensional problems

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Abstract

An integral equations method for a three-dimensional crack in a finite or infinite body is achieved by means of Kupradze potentials. Surface and through cracks can be studied according to this approach with only the assumption that the body has a linear, elastic, homogeneous and isotropic behavior. Both singular surface integrals and line integrals appear in the derived equations. For surface and through cracks, the line integral is taken on a part of the crack boundary. The use of our integral equations to the particular problem of an embedded plane crack leads to those formulated by Bui. Another application is devoted to a through crack in a circular cylinder.

1. Introduction

For plane problems, linear fracture mechanics permits predicting crack behavior from the knowledge of stress intensity factors. We expect these parameters are always relevant to three-dimensional cracks; unfortunately we cannot derive them for general cases. Complexity results, of course, from the three-dimensional feature of the problem, from the connexion with boundaries of the solid when it is finite and from the fact that stress intensity factors vary all along the crack front. Except for the particular case of the penny shaped crack studied by Sneddon [1] one does not know closed solutions to more general problems.

Present trends are turning towards the use of integral equations and, for many authors, Somigliana formula is considered as a starting point. This method enabled Cruse [2] to write integral equations for an uncracked body. The problem of a cracked body can be tackled with the help of Kupradze elastic potentials [3], also known as Bashelishvili potentials.

These elastic potentials were applied by Bui [4] in order to solve the problem of an arbitrarily shaped plane crack embedded in an infinite body. He obtained its solution by the help of the sum of two potentials: the first is a simple layer and the second is a double-layer of the second kind. The use of these potentials and symmetry considerations allowed Putot [5] to deal with the problem of a plane surface crack perpendicular to the free plane boundary of a semi-infinite body.

After transformation of Somigliana formula, V. and J. Slâdek [6] achieved the derivation of the stress vector at any point of the body in terms of the displacement discontinuity resulting from a three-dimensional crack embedded in an infinite body. Properties of Kupradze potentials [7] enable them to pass to the limit and explain the stress value at any point of the crack surface; this is always given in terms of displacement discontinuities.

First, we defined the displacement field inside a cracked body by means of a double-layer Kupradze potential of the first kind, then we proved that stress vectors can be explained in terms of a density function defined on an open set including crack surface. The passing to the limit enables us to derive stress vector on crack surfaces.

Next we showed that only the partial derivatives of the unknown densities restriction on crack surfaces, with respect to suitable variables, are involved in the integral equation. This result confirms the idea that the problem is well formulated.

We provided another confirmation of our approach in solving the problem of a plane crack as dealt with by Bui. By means of a similar transformation to that formerly proposed by V. and J. Slâdek we succeeded in dealing with embedded crack problems as well as transverse crack ones.

The last step of this work is devoted to an application of our results to the problem of a plane through crack lying in the cross section of a thick tube. This points out the contribution, in the integral equation, of the line integral that does not appear in the previous problems concerning embedded cracks.

Integral equations derived below can be applied to any finite or infinite body containing a three dimensional crack; it can be a surface or a through crack. Linear elastic behavior of the medium is the only assumption required.

2. Basic notations

The elastic body under consideration is denoted by D, characterized by either the Lamé constants λ, μ or the Young's modulus E and the Poisson's ratio ν. Its exterior boundary is denoted by ∂D.

The crack S is considered as a geometric surface, and not as the union of its upper and lower faces S^+ and S^-. It is assumed that S is a Liapunov surface, i.e. it belongs to the class $V^{1,\alpha}$, $0 < \alpha \leqslant 1$. The boundary of the crack surface S is denoted by ∂S, which is not included in S, so that the interior of S is S itself:

$$\mathring{S} = S \backslash \partial S = S$$

$\boldsymbol{e}_i (i = 1, 2, 3)$ is a unit vector of a fixed Cartesian coordinates system in the Euclidean space E_3. A point of E_3 and the corresponding radius-vector are denoted by the same symbol. $\boldsymbol{n}_x$, $\boldsymbol{n}_z$ designate normals at any points $\boldsymbol{x}$, z of D, $\boldsymbol{y}$ is any point of S, and $\boldsymbol{x}_0$ is a particular one. The notations $\boldsymbol{n}(\boldsymbol{x})$, $\boldsymbol{n}(z)$ will not be used, since an infinite number of normals can be chosen at each point, the normal is not a function of this latter. Note, though, that for convenience reasons, one may encounter the i-component of $\boldsymbol{n}_x$ denoted by $n_i(\boldsymbol{x})$, instead of n_{xi}. Moreover, an orientation of S being chosen once and for all, $\boldsymbol{n}_y$ or $\boldsymbol{n}_{x_0}$ will not refer to any normals of S at the point y, $\boldsymbol{x}_0$ respectively, but those orienting the surface S.

The displacement, which is a function of $\boldsymbol{x}$, is denoted by $\boldsymbol{u}(\boldsymbol{x})$. At any point $\boldsymbol{x}$ the stress tensor is $\overline{\overline{\Sigma}}(\boldsymbol{x})$ and the stress vector, with respect to the normal $\boldsymbol{n}_x$, is $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}_x)$. For a point $\boldsymbol{x}_0 \in S$, $\boldsymbol{t}(\boldsymbol{x}_0, \boldsymbol{n}_{x_0})$ is understood as a limit value: lim $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}_x)$ as $\boldsymbol{x} \to \boldsymbol{x}_0$, $\boldsymbol{n}_x \to \boldsymbol{n}_{x_0}$, the normal $\boldsymbol{n}_{x_0}$ being well-defined by the orientation of S.

An implicit sum is implied on any repeated indices. Thus, let f and ψ be some differentiable functions in D, we have:

$$\text{grad } f = f_{,i} \cdot \boldsymbol{e}_i$$

$$\text{rot } \psi = -\epsilon_{ijk} \cdot \psi_{j,k} \cdot \boldsymbol{e}_i$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}n} = \psi_{i,j} n_j \boldsymbol{e}_i$$

Defining the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ by:

$$\forall \boldsymbol{c}, \quad (\boldsymbol{a} \otimes \boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.(\boldsymbol{b}.\boldsymbol{c}),$$

we have:

$$\text{grad } \boldsymbol{\psi} = \psi_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$\overline{\overline{T}}^T$, the transposed tensor of $\overline{\overline{T}}$ is defined by:

$$\forall i, \ j: (\overline{\overline{T}}^T)_{ij} = (\overline{\overline{T}})_{ji}.$$

$\overline{\overline{I}}$ denotes the unit tensor:

$$\forall i, \ j: (\overline{\overline{I}})_{ij} = \delta_{ij},$$

where:

$$\delta_{ij} = \left| \begin{array}{ll} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{array} \right. \quad \text{and } \epsilon_{ijk} = (\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{e}_i(\boldsymbol{e}_j \Lambda \boldsymbol{e}_k)$$

The symbols pv $\int$ or $\int^*$ indicate that the associated integral must be understood in the sense of the Cauchy principal value.

Let f denote a function of two variables $\boldsymbol{x}$, $\boldsymbol{z}$. Unless stated otherwise, $f_{,i}$ denotes the partial derivatives of f with respect to z_i:

$$f_{,i} \equiv \frac{\partial f}{\partial z_i}$$

and $f_{,i}(\boldsymbol{y} \in S)$ designates the value of the derivative of f, with respect to z_i, at a point $\boldsymbol{y} \in S$:

$$f_{,i}(\boldsymbol{y} \in S) \equiv \left. \frac{\partial f(\boldsymbol{z})}{\partial z_i} \right|_{\boldsymbol{z} = \boldsymbol{y} \in S}$$

When partial derivatives with respect to x_i are involved, it will be explicitly mentioned.

3. Definition of the auxiliary problem

Let us consider an elastic body D under arbitrary loading, containing a crack S unloaded on its faces. Without this crack, the state of stress in the body would be $\overline{\overline{\Sigma}}^0$, and the two faces of S would remain in contact with one another under internal forces resulting from $\overline{\overline{\Sigma}}^0$: $\boldsymbol{t}^0(\boldsymbol{x}_0, \boldsymbol{n}_{x_0})$, $\boldsymbol{x}_0 \in S$. These internal forces are computed on the *uncracked* finite body subjected to the aforementioned loading.

In order to obtain unloaded crack faces, it is necessary to add on the crack the stresses $-\boldsymbol{t}^0(\boldsymbol{x}_0, \boldsymbol{n}_{x_0})$, opposite to those resulting from the initial stress state $\overline{\overline{\Sigma}}^0$, whereas, the external loading applied on ∂D, is forced to zero. This loading on the crack faces will generate another state of stress in the body, denoted by $\overline{\overline{\Sigma}}^1$.

Thus, the solution to the problem of a cracked body can be obtained by superposition of $\overline{\overline{\Sigma}}^1$ on $\overline{\overline{\Sigma}}^0$:

$$\overline{\overline{\Sigma}}(\boldsymbol{x}) = \overline{\overline{\Sigma}}^0(\boldsymbol{x}) + \overline{\overline{\Sigma}}^1(\boldsymbol{x})$$

where $\overline{\overline{\Sigma}}^0(\boldsymbol{x})$ is a regular stress tensor and $\overline{\overline{\Sigma}}^1(\boldsymbol{x})$ is a singular one in the vicinity of the crack edge ∂S (Fig. 1).

This leads us to consider what is called *the auxiliary problem* stated as the problem of a crack under arbitrary loadings. The boundary conditions are:

$$\forall \boldsymbol{x} \in S: \quad \lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}, \ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}_x) = -\boldsymbol{t}^0(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) \tag{1}$$

$$\forall \boldsymbol{x} \in \partial D: \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}_x) = \boldsymbol{0} \tag{2}$$

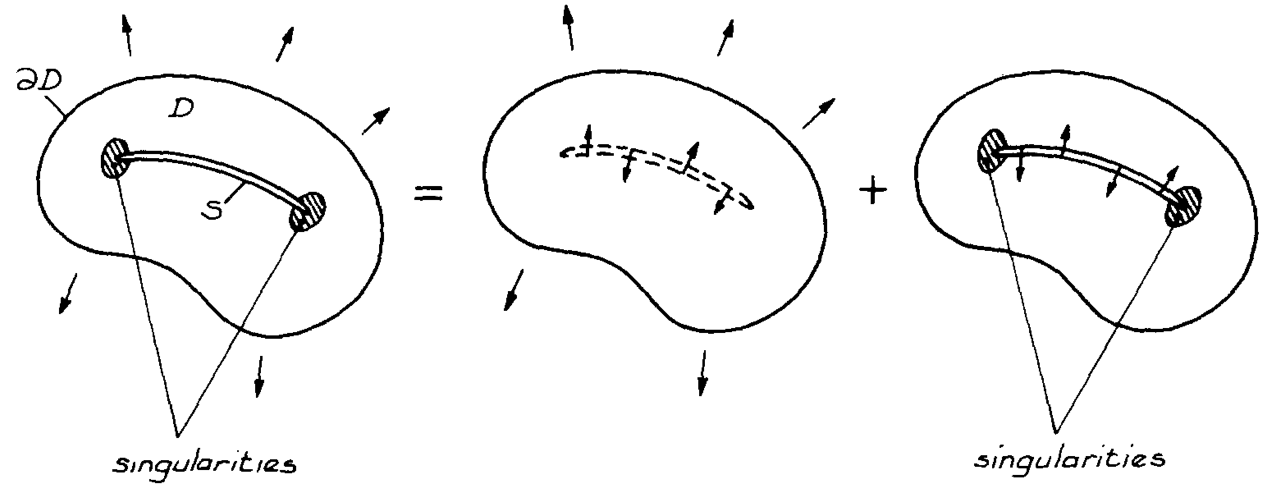

Figure 1. Analysis of the problem of a cracked body.

In fact, one begins to obtain a solution to the problem with the boundary conditions (1) and another one slightly different from (2):

$$\forall \boldsymbol{x}, \|\boldsymbol{x}\| \to \infty: \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}_x) = \boldsymbol{0} \tag{2bis}$$

The corresponding solution is rather satisfactory in the case of a little crack embedded in a large body. Nevertheless, if the crack is not sufficiently far away from the boundary ∂D of the body (especially in the case of surface and through cracks) use must be made of an appropriate method for satisfying the boundary condition (2). This method will be illustrated in the example at the end of this paper.

4. Integral equations for the auxiliary problem

1) Definition of the displacement field

Let us represent the displacement field by means of the double-layer potential of the first kind of Kupradze [7]:

$$\forall \boldsymbol{x} \in E_3 (\text{instead of } D)\text{: } \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{W}_I(\boldsymbol{x}) = \int_S \bar{\bar{T}}^T(\boldsymbol{y} - \boldsymbol{x}, \boldsymbol{n}_y)\boldsymbol{\psi}(\boldsymbol{y})\, \mathrm{d}S_y \tag{3}$$

where the density $\boldsymbol{\psi}$ is a vector function of three variables $z = (z_1, z_2, z_3)$ defined on an open set ϑ containing S. Its restriction on $S \boldsymbol{\psi}_{|S}(z) = \boldsymbol{\psi}(\boldsymbol{y} \in S)$ satisfying the condition: $\boldsymbol{\psi}_{|S} \in C^{1,\alpha}(S)$, $0 < \alpha \leqslant 1$, is the unknown of the problem.

It is proved in [7] that $\boldsymbol{u}(\boldsymbol{x})$ thus defined satisfies the homogeneous static Navier's equation:

$$\mathscr{L}\boldsymbol{u} \equiv \mu\Delta\boldsymbol{u} + (\lambda + \mu) \text{ grad div } \boldsymbol{u} = \boldsymbol{0}, \quad \forall \boldsymbol{x} \in E_3 \setminus S \tag{4}$$

It must be stated that the stress state generated by this displacement field satisfies the boundary condition (1), the condition (2bis) being identically satisfied.

Following [1], we have a relation on limit behaviour of the displacement:

$$\boldsymbol{u}(\boldsymbol{x}_0^\pm) = \pm\tfrac{1}{2}\boldsymbol{\psi}(\boldsymbol{x}_0) + pv\int_S \bar{\bar{T}}^T(\boldsymbol{y} - \boldsymbol{x}_0, \boldsymbol{n}_y)\boldsymbol{\psi}(\boldsymbol{y})\, \mathrm{d}S_y, \quad \forall \boldsymbol{x}_0 \in S \tag{5}$$

where:

$$\begin{aligned} \bar{\bar{T}}(z - \boldsymbol{x}, \boldsymbol{n}_z) = {} & \frac{2\mu}{8\pi(\lambda + 2\mu)r^2}\left[\boldsymbol{n}_z \otimes \boldsymbol{e}_r - \boldsymbol{e}_r \otimes \boldsymbol{n}_z - (\boldsymbol{e}_r.\boldsymbol{n}_z)\bar{\bar{I}}\right] \\ & - \frac{6(\lambda + \mu)}{8\pi(\lambda + 2\mu)} \cdot \frac{\boldsymbol{e}_r.\boldsymbol{n}_z}{r^2}\boldsymbol{e}_r \otimes \boldsymbol{e}_r \end{aligned} \tag{6}$$

$$r = r(z - x), \qquad e_r = e_r(z - x),$$
$$\bar{\bar{T}}(y - x, n_y) = \bar{\bar{T}}(z - x, n_z)\,|_{z = y \in S}$$

Using (5), we obtain the displacement discontinuity on the crack:

$$[u(x_0)] \equiv u(x_0^+) - u(x_0^-) = \psi(x_0), \quad \forall x_0 \in S \tag{7}$$

This discontinuity is thus directly related to the unknown density $\psi(y \in S)$. Knowing $\psi(y \in S)$ will allow us to calculate, in particular, the stress intensity factors.

2) The state of stress

Since the boundary conditions are expressed in terms of stresses, we have to derive the expression of the stress from the displacement. According to (3), we have:

$$\forall x \in E_3 : t(x, n_x) = \mathscr{T}(\partial_x, n_x) u(x) = \mathscr{T}(\partial_x, n_x) \int_S \bar{\bar{T}}^T(y - x, n_y) \psi(y)\, \mathrm{d}S_y \tag{8}$$

where $\mathscr{T}(\partial_x, n_x)$ is the stress operator:

$$\mathscr{T}(\partial_x, n_x) = 2\mu \frac{\mathrm{d}}{\mathrm{d}n_x} + \mu n_x \Lambda \text{ rot} + \lambda n_x \text{ div} = c_{ijkl}.n_j(x).\frac{\partial}{\partial x_l} \tag{9}$$

where:

$$c_{ijkl} = \lambda \delta_{ij}.\delta_{kl} + \mu(\delta_{ik}.\delta_{jl} + \delta_{il}.\delta_{kj}) \tag{10}$$

Let the Green's tensor of the Navier's operator be $\bar{\bar{E}}(z - x)$ (also identified as Kelvin-Somigliana tensor):

$$\bar{\bar{E}}(z - x) = \frac{1}{8\pi\mu(\lambda + 2\mu)} \left[(\lambda + 3\mu)\bar{\bar{I}} + (\lambda + \mu) e_r \otimes e_r\right].\frac{1}{r} \tag{11}$$

Where:

$$e_r = e_r(z - x), \quad r = r(z - x)$$

By denoting:

$$U(e_i, z - x) = \bar{\bar{E}}(z - x).e_i$$

We have [7] (see notations of (4)):

$$\mathscr{L} U(e_i, z - x) = -\delta_{R^3}(z - x).e_i \tag{12}$$

where $\delta_{R^3}(z - x)$ denotes the Dirac measure concentrated at the point z.

On account of the equality:

$$\bar{\bar{T}}(z - x, n_z) = \mathscr{T}(\partial_z, n_z) \bar{\bar{E}}(z - x),$$

the stress vector at a point x, with respect to the normal n_x can be rewritten in the form:

$$t(x, n_x) = \mathscr{T}(\partial_x, n_x) \int_S \left[\mathscr{T}(\partial_z, n_z) \bar{\bar{E}}(z - x)\right]^T|_{z=y}.\psi(y)\, \mathrm{d}S_y \tag{13}$$

i.e., for the l-component, using:

$$\frac{\partial E_{ij}}{\partial x_k}(z - x) = -\frac{\partial E_{ij}}{\partial z_k}(z - x)$$

$$t_l(x, n_x) = \mathscr{T}_{lj}(\partial_x, n_x) \int_S \left[\mathscr{T}_{ik}(\partial_z, n_z) E_{kj}(z - x)\right]|_{z=y} \psi_i(y)\, \mathrm{d}S_y$$

$$= -c_{lpjr} c_{iskt}.n_p(x) \int_S \frac{\partial}{\partial z_r} \frac{\partial E_{kj}}{\partial z_t}(y - x).n_s(y).\psi_i(y)\, \mathrm{d}S_y \tag{14}$$

where:

$$\frac{\partial}{\partial z_r}\frac{\partial E_{kj}}{\partial z_t}(y-x)\equiv\frac{\partial}{\partial z_r}\frac{\partial E_{kj}(z-x)}{\partial z_t}\bigg|_{z=y\in S}$$

Since E_{kj} has a point singularity of the type r^{-1}, the kernel of integral (14) has a singularity of the type r^{-3}. This prevents us from applying the theorems on the limit behaviour of double-layer potentials. Therefore it is necessary to perform some additional transformations so as to obtain afterwards a kernel of the type r^{-2}. This will be achieved by using the Stokes theorem:

$$\int_S \operatorname{rot}_z \boldsymbol{a}(y).\boldsymbol{n}_y\, \mathrm{d}S_y = \int_{\partial S}\boldsymbol{a}(y)\, \mathrm{d}\boldsymbol{l}_y$$

where rot_z implies that the derivatives must be performed with respect to the variable z_i:

$$\operatorname{rot}_z \boldsymbol{a}(y)\equiv \operatorname{rot}_z \boldsymbol{a}(z)\,|_{z=y\in S}$$

and $\mathrm{d}\boldsymbol{l}_y$ is the contour element vector:

$$\mathrm{d}\boldsymbol{l}_y=\mathrm{d}\,y_i.\boldsymbol{e}_i.$$

Setting:

$$\boldsymbol{a}(z)=f(z).\boldsymbol{a}_0$$

where $\boldsymbol{a}_0$ is some arbitrary constant vector, we obtain:

$$\int_S \operatorname{grad}_z f\wedge \boldsymbol{n}_y\, \mathrm{d}S_y=-\oint_{\partial S} f\, \mathrm{d}l.$$

i.e., for the m-component:

$$\int_S \epsilon_{mjk} f_{,j}.n_k(y)\, \mathrm{d}S_y=-\oint_{\partial S} f\,\mathrm{d}\,y_m \tag{15}$$

Multiplying both sides of (15) by ϵ_{msr}, and taking into account:

$$\epsilon_{msr}.\epsilon_{mjk}=\delta_{sj}.\delta_{rk}-\delta_{sk}.\delta_{rj}$$

we arrive at the following relation:

$$\oint_{\partial S}\epsilon_{msr}.f\,\mathrm{d}\,y_m=\int_S (f_{,r}n_s-f_{,s}n_r)\,\mathrm{d}S \tag{16}$$

Setting here:

$$f(z)=\psi_i(z).E_{kj,t}(z-x),$$

we obtain:

$$\int_S \psi_i n_s E_{kj,tr}\,\mathrm{d}S=\int_S \kappa^i_{rs}.E_{kj,t}\,\mathrm{d}S+\int_S \psi_i n_r E_{kj,st}\,\mathrm{d}S+\oint_{\partial S}\epsilon_{msr}\psi_i E_{kj,t}\,\mathrm{d}\,y_m \tag{17}$$

where $\kappa^{\mathrm{i}}_{\mathrm{rs}}$ is defined by:

$$\kappa^i_{rs}(y)=n_r(y)\psi_{i,s}(y)-n_s(y)\psi_{i,r}(y)$$

$$\equiv n_r(y)\frac{\partial\psi_i(z)}{\partial z_s}\bigg|_{z=y\in S}-n_s(y)\frac{\partial\psi_i(z)}{\partial z_r}\bigg|_{z=y\in S} \tag{18}$$

Substituting (17) in (14) gives:

$$t_l(\boldsymbol{x},\ \boldsymbol{n}_x) = -c_{lpjr}c_{iskt}n_p(\boldsymbol{x})\int_S \kappa^i_{rs}E_{kj,t}(\boldsymbol{y}-\boldsymbol{x})\ \mathrm{d}S$$

$$-c_{lpjr}c_{iskt}n_p(\boldsymbol{x})\int_S \psi_i n_r E_{kj,st}\ \mathrm{d}S$$

$$-c_{lpjr}c_{iskt}n_p(\boldsymbol{x})\oint_{\partial S} \epsilon_{mst}\psi_i E_{kj,t}\ \mathrm{d}y_m.$$

Now in virtue of (10) and (12):

$$c_{iskt}E_{kj,st} = \mu E_{ij,ss} + (\lambda+\mu)E_{kj,ik} = \boldsymbol{e}_j.\mathscr{L}\boldsymbol{U}(\boldsymbol{e}_i,\ \boldsymbol{y}-\boldsymbol{x}) = -\delta_{ij}.\delta_{R^3}(\boldsymbol{y}-\boldsymbol{x}) = 0$$

since $\boldsymbol{x}\notin S$, we eventually obtain:

$$t_l(\boldsymbol{x},\ \boldsymbol{n}_x) = -c_{lpjr}c_{iskt}n_p(\boldsymbol{x})\int_S \kappa^i_{rs}.\frac{\partial E_{kj}}{\partial z_t}(\boldsymbol{y}-\boldsymbol{x})\ \mathrm{d}S_y$$

$$+c_{lpjr}c_{iskt}n_p(\boldsymbol{x})\oint_{\partial S} \epsilon_{mrs}\psi_i E_{kj,t}\ \mathrm{d}y_m$$

which gives, after performing all summations:

$$\begin{aligned}
t_l(\boldsymbol{x},\ \boldsymbol{n}_x) = {} & \frac{\mu n_p(\boldsymbol{x})}{8\pi(1-\nu)}\int_S \frac{1}{r^2}\Big\{4\nu\delta_{lp}\kappa^i_{ik}r_{,k} \\
& + (1-2\nu)\left[\kappa^i_{ip}r_{,l} + \kappa^i_{il}r_{,p} + r_{,k}\left(\kappa^l_{pk}+\kappa^p_{lk}\right)\right] \\
& + 3r_{,i}r_{,k}\left(\kappa^i_{pk}r_{,l}+\kappa^i_{lk}r_{,p}\right)\Big\}\ \mathrm{d}S_y \\
& - \frac{\mu n_p(\boldsymbol{x})}{8\pi(1-\nu)}\oint_{\partial S}\frac{\psi_i}{r^2}\Big\{4\nu\delta_{lp}\epsilon_{mik}r_{,k} + (1-2\nu) \\
& \times\left[\epsilon_{mip}r_{,l} + \epsilon_{mil}r_{,p} + r_{,k}\left(\epsilon_{mpk}\delta_{il}+\epsilon_{mlk}\delta_{ip}\right)\right] \\
& + 3r_{,i}r_{,k}\left(\epsilon_{mpk}r_{,l}+\epsilon_{mlk}r_{,p}\right)\Big\}\ \mathrm{d}y_m
\end{aligned} \tag{19}$$

where:

$$r = r(\boldsymbol{y}-\boldsymbol{x}),\quad r_{,i} \equiv \left.\frac{\partial r(\boldsymbol{z}-\boldsymbol{x})}{\partial z_i}\right|_{\boldsymbol{z}=\boldsymbol{y}\in S}$$

Thus, the stress vector is written as a sum of surface and line integrals, the kernel of surface integral is singular as r^{-2}. Next we will pass to the limit as $\boldsymbol{x}\to\boldsymbol{x}_0^{\pm}$, the limit behaviour of double-layer potential being now available. On the other hand, when passing to the limit, there is no singularity in the line integral, since we have assumed the surface S does not include its boundary ∂S (see §2 – Basic notations), so that r^{-2} in the line integral remains bounded when $\boldsymbol{x}$ tends to $\boldsymbol{x}_0\in S$. However, for $\boldsymbol{x}_0$ close by the contour ∂S, numerical difficulties should be expected.

Starting from another point of view, V. Sladek and J. Sládek [6] obtained similar equations where the displacement discontinuity $[\boldsymbol{u}]$ was involved instead of ψ. Despite the equality (7), there are more difficulties in formulating surface or through crack problems in terms of $[\boldsymbol{u}]$. This will be well illustrated in the example by the end of this paper. For embedded crack problems, these two points of view are equivalent. Moreover, in [6], the crack S is considered as a closed surface resulting from the union of its upper and lower faces S^+ and S^-, so the line integral taken on the boundary ∂S vanishes. On the contrary

S, herein considered, is regarded as a geometric surface through which the displacement field is subjected to a discontinuity.

For an embedded crack we have:

$$[\boldsymbol{u}(y)] \equiv \boldsymbol{u}(y^+) - \boldsymbol{u}(y^-) \equiv \mathbf{0}, \quad \forall y \in \partial S,$$

which is equivalent to, by virtue of (7):

$$\psi(y) = \mathbf{0}, \quad \forall y \in \partial S,$$

thus the line integral vanishes, and this result is consistent with Slàdek's equations. On the other hand, for a surface or through crack, we have:

$$\exists y \in \partial S : [\boldsymbol{u}(y)] \neq \mathbf{0}.$$

The line integral does not vanish on a portion L included in ∂S (L may be the union of separate arcs of ∂S), and it is reduced to a line integral taken along L. As mentioned before, the kernel of the line integral is not singular, we shall denote it for simplicity:

$$\int_L \frac{\psi_i(y) \cdot f^i_{lmp}(y - x)}{r^2(y - x)} \mathrm{d} y_m$$

5. Integral equations

Formula (19) will allow us to express the boundary condition (1) in terms of stresses. For this purpose, the limit of the l-component of stress $t_l(\boldsymbol{x}, \boldsymbol{n}_x)$ as $\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}$ and $\boldsymbol{n}_x = \boldsymbol{n}_{x_0}$, $\boldsymbol{x}_0 \in S$, is investigated (Fig. 2).

As was mentioned previously in §2 – Basic notations, S is a Liapunov surface, i.e. belonging to the class $V^{1,\alpha}$, $0 < \alpha \leqslant 1$. Moreover, $\psi_{|S}$ was assumed to belong to the class

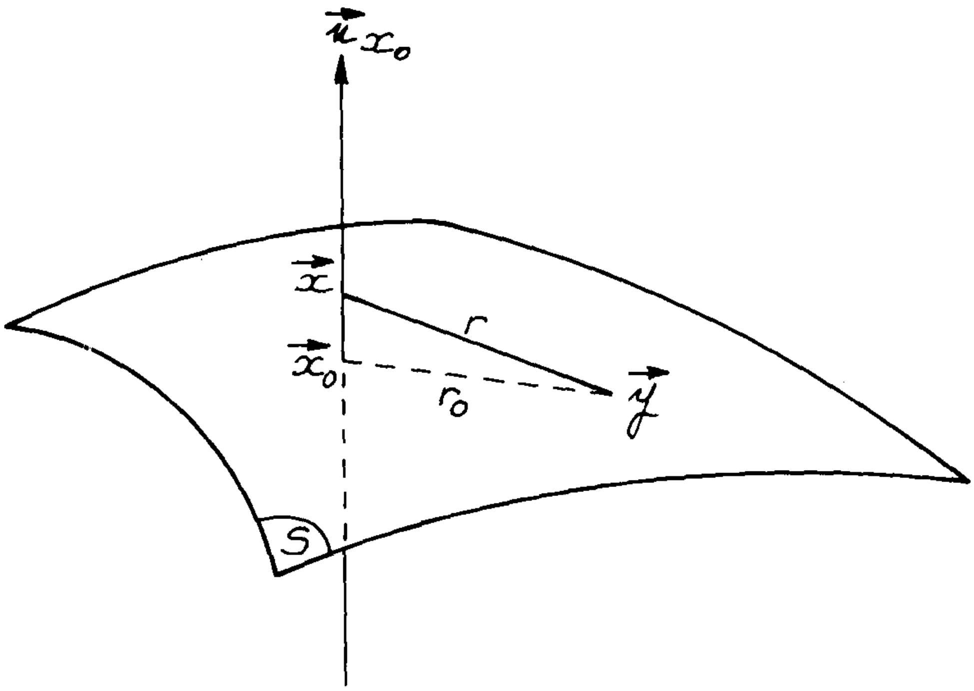

Figure 2. Case when $\boldsymbol{x} \to \boldsymbol{x}_0^+$ and $\boldsymbol{n}_x = \boldsymbol{n}_{x_0}$

$r = r(y - \boldsymbol{x})$, $r_0 = r(y - \boldsymbol{x}_0)$

$C^{1,\alpha}(S) \subset C^{0,\alpha}(S)$. Then one can prove the following formulae, valid for any point $\boldsymbol{x}_0 \in S$:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{\boldsymbol{e}_r(\boldsymbol{y}-\boldsymbol{x}) \otimes \boldsymbol{n}_y}{r^2(\boldsymbol{y}-\boldsymbol{x})} \psi(\boldsymbol{y}) \, \mathrm{d}S_y$$

$$= \mp 2\pi \left(\boldsymbol{n}_{x_0} \otimes \boldsymbol{n}_{x_0}\right) \psi(\boldsymbol{x}_0) + pv \int_S \frac{\boldsymbol{e}_r(\boldsymbol{y}-\boldsymbol{x}_0) \otimes \boldsymbol{n}_y}{r^2(\boldsymbol{y}-\boldsymbol{x}_0)} . \psi(\boldsymbol{y}) \, \mathrm{d}S_y \tag{20a}$$

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{\boldsymbol{e}_r(\boldsymbol{y}-\boldsymbol{x}) . \boldsymbol{n}_y}{r^2(\boldsymbol{y}-\boldsymbol{x})} \psi(\boldsymbol{y}) \, \mathrm{d}S_y$$

$$= \mp 2\pi \psi(\boldsymbol{x}_0) + pv \int_S \frac{\boldsymbol{e}_r(\boldsymbol{y}-\boldsymbol{x}_0) . \boldsymbol{n}_y}{r^2(\boldsymbol{y}-\boldsymbol{x}_0)} . \psi(\boldsymbol{y}) \, \mathrm{d}S_y \tag{20b}$$

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{n_l r_{,p}}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y$$

$$= \mp \frac{2\pi}{3} \left[n_l (\boldsymbol{e}_p \otimes \boldsymbol{n}_y + \boldsymbol{n}_y \otimes \boldsymbol{e}_p) + n_l n_p \left(\bar{\bar{I}} - 2\boldsymbol{n}_y \otimes \boldsymbol{n}_y \right) \right] \psi(\boldsymbol{x}_0)$$

$$+ pv \int_S \frac{n_l r_{,p}}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y \tag{20c}$$

It should be noticed that every function occuring in the left-hand sides of (20) is expressed in terms of $\boldsymbol{y}$ or $(\boldsymbol{y}-\boldsymbol{x})$, whereas that in the right-hand side in terms of $\boldsymbol{y}$ or $(\boldsymbol{y}-\boldsymbol{x}_0)$. One also remarks the discontinuity when passing to the limit as $\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}$.

From (20c), we obtain by changing the indices l, p, a symmetric equation:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{n_l r_{,p} + n_p r_{,l}}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y$$

$$= \mp \frac{2\pi}{3} \left[(n_l \boldsymbol{e}_p + n_p \boldsymbol{e}_l) \otimes \boldsymbol{n}_y + \boldsymbol{n}_y \otimes (n_l \boldsymbol{e}_p + n_p \boldsymbol{e}_l) + 2 n_l n_p \left(\bar{\bar{I}} - 2\boldsymbol{n}_y \otimes \boldsymbol{n}_y \right) \right] \psi(\boldsymbol{x}_0)$$

$$+ pv \int_S \frac{n_l r_{,p} + n_p r_{,l}}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y \tag{20d}$$

Moreover, setting $l = p$ in (20c), we have:

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{\boldsymbol{e}_r . \boldsymbol{n}_y}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y$$

$$= \mp \frac{2\pi}{3} \psi(\boldsymbol{x}_0) + pv \int_S \frac{\boldsymbol{e}_r . \boldsymbol{n}_y}{r^2} (\boldsymbol{e}_r \otimes \boldsymbol{e}_r) \psi(\boldsymbol{y}) \, \mathrm{d}S_y \tag{20e}$$

Using (20), we easily obtain:

$$n_p(\boldsymbol{x}_0) \delta_{lp} \lim_{\boldsymbol{x} \to \boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x = \boldsymbol{n}_{x_0}} \int_S \frac{\kappa^i_{ik} r_{,k}}{r^2} \mathrm{d}S_y$$

$$= \mp 2\pi \left(n_l n_i n_k \psi_{i,k} - n_l \psi_{i,i} \right) + n_l(\boldsymbol{x}_0) . pv \int_S \frac{\kappa^i_{ik} . r_{,k}}{r^2} \mathrm{d}S_y \tag{21a}$$

$$n_p(\boldsymbol{x}_0) \lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x=\boldsymbol{n}_{x_0}} \int_S \frac{1}{r^2}\left[\kappa^i_{ip} r_{,l} + \kappa^i_{il} r_{,p} + r_{,k}\left(\kappa^l_{pk} + \kappa^p_{lk}\right)\right] \mathrm{d}S_y$$

$$= \mp 2\pi\left(n_l n_p n_i \psi_{i,p} - n_l \psi_{i,i}\right) + n_p(\boldsymbol{x}_0)$$

$$\times pv \int_S \frac{1}{r^2}\left[\kappa^i_{ip} r_{,l} + \kappa^i_{il} r_{,p} + r_{,k}\left(\kappa^l_{pk} + \kappa^p_{lk}\right)\right] \mathrm{d}S_y \tag{21b}$$

$$n_p(\boldsymbol{x}_0) \lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x=\boldsymbol{n}_{x_0}} \int_S \frac{1}{r^2} r_{,i} r_{,k}\left(\kappa^i_{pk} r_{,l} + \kappa^i_{lk} r_{,p}\right) \mathrm{d}S_y$$

$$= \mp \frac{2\pi}{3}\left(-n_p \psi_{l,p} - n_p \psi_{p,l} - 2 n_l n_i n_p \psi_{i,p} + n_k \psi_{l,k} + n_i \psi_{i,l} + 2 n_l \psi_{i,i}\right)$$

$$+ n_p(\boldsymbol{x}_0).pv \int_S \frac{1}{r^2} r_{,i} r_{,k}\left(\kappa^i_{pk} r_{,l} + \kappa^i_{lk} r_{,p}\right) \mathrm{d}S_y \tag{21c}$$

We recall that functions in the left-hand side of (21c) are reckoned at y or $(y - \boldsymbol{x})$, whereas those in the right-hand side at y or $(y - \boldsymbol{x}_0)$. On account of (21), the limit of the stress vector $t_l(\boldsymbol{x}, \boldsymbol{n}_x)$ as $\boldsymbol{x} \to \boldsymbol{x}_0^{\pm}$, $\boldsymbol{n}_x = \boldsymbol{n}_{x_0}$, can be written in the form:

$$\lim_{\boldsymbol{x}\to\boldsymbol{x}_0^{\pm},\ \boldsymbol{n}_x=\boldsymbol{n}_{x_0}} t_l(\boldsymbol{x}, \boldsymbol{n}_x) = \mp[\ldots] + \frac{\mu n_p(\boldsymbol{x}_0)}{8\pi(1-\nu)} pv \int_S [\ldots]\, \mathrm{d}S_y - \frac{\mu n_p(\boldsymbol{x}_0)}{8\pi(1-\nu)}.$$

$$\int_L \frac{\psi_i(y) f^i_{lmp}(y - \boldsymbol{x}_0)}{r^2(y - \boldsymbol{x}_0)} \mathrm{d}y_m$$

One can easily verify that the first bracket is identically zero, so that the limits as $\boldsymbol{x} \to \boldsymbol{x}_0^+$ and $\boldsymbol{x} \to \boldsymbol{x}_0^-$ coincide. This "continuity" is predicted by the Liapunov-Tauber theorem (see for instance [7] or [8] as well). It should be noticed that the line integral on L is not singular. We eventually arrive at the following formula:

$$t_l(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) = \frac{\mu n_p(\boldsymbol{x}_0)}{8\pi(1-\nu)} pv \int_S \frac{1}{r^2}\left\{4\nu \delta_{lp} \kappa^i_{ik} r_{,k} + (1 - 2\nu)\right.$$

$$\times \left[\kappa^i_{ip} r_{,l} + \kappa^i_{il} r_{,p} + r_{,k}\left(\kappa^l_{pk} + \kappa^p_{lk}\right)\right] + 3 r_{,i} r_{,k}\left.\left(\kappa^i_{pk} r_{,l} + \kappa^i_{lk} r_{,p}\right)\right\}$$

$$\times \mathrm{d}S_y - \frac{\mu n_p(\boldsymbol{x}_0)}{8\pi(1-\nu)} \int_L \frac{\psi_i(y).f^i_{lmp}(y - \boldsymbol{x}_0)}{r^2(y - \boldsymbol{x}_0)} \mathrm{d}y_m \tag{22}$$

where $t_l(\boldsymbol{x}_0, \boldsymbol{n}_{x_0})$ is understood in the sense of a limit, and:

$$\kappa^i_{jk}(y) = n_j(y).\psi_{i,k}(y) - n_k(y)\psi_{i,j}(y)$$

$$r = r(y - \boldsymbol{x}_0) = \| y - \boldsymbol{x}_0 \|$$

$$r_{,i} \equiv \left.\frac{\partial r(z - \boldsymbol{x})}{\partial z_i}\right|_{\substack{z=y\in S\\ \boldsymbol{x}=x0\in S}} = \left.\frac{\partial r(z - \boldsymbol{x}_0)}{\partial z_i}\right|_{z=y\in S}$$

For calculations in curvilinear coordinates (e.g. in cylindrical or spherical coordinates) it should be interesting to write (22) in the tensor form. Without more details, we give the final result:

$$
\begin{aligned}
t(x_0, n_{x_0}) = {} & \frac{\mu}{8\pi(1-\nu)} pv \int_S \frac{1}{r^2} \Big\{ 4\nu\big(n_y \operatorname{grad}\psi e_r - (n_y e_r) \operatorname{div} \psi\big) n_{x_0} \\
& + \Big[(1-2\nu)\big(n_y \operatorname{grad}\psi n_{x_0} - (n_y n_{x_0}) \operatorname{div} \psi\big) + 3(n_y n_{x_0}) e_r \operatorname{grad}\psi e_r \\
& - 3(n_y e_r)(e_r \operatorname{grad}\psi n_{x_0})\Big] e_r + \Big[(1-2\nu)\big(n_{x_0} \operatorname{grad}\psi e_r - (n_{x_0} e_r) \operatorname{div} \psi\big) \\
& + 3(n_{x_0} e_r) e_r \operatorname{grad}\psi e_r\Big] n_y + (1-2\nu)\Big[(n_{x_0} e_r) \operatorname{grad}^T \psi n_y \\
& + (n_y n_{x_0}) \operatorname{grad}\psi e_r - (n_y e_r) \operatorname{grad}\psi n_{x_0} - (n_y e_r) \operatorname{grad}^T \psi n_{x_0}\Big] \\
& - 3(n_{x_0} e_r)(n_y e_r) \operatorname{grad}^T \psi e_r \Big\} \, \mathrm{d}S_y \\
& - \frac{\mu}{8\pi(1-\nu)} \int_L \frac{1}{r^2} \Big\{ 4\nu\big[\psi(e_r \Lambda \, \mathrm{d}l)\big] n_{x_0} + \Big\{(1-2\nu)\big[(\psi \Lambda n_{x_0}) \, \mathrm{d}l\big] \\
& + 3\big[n_{x_0}(e_r \Lambda \, \mathrm{d}l)\big](e_r \psi)\Big\} e_r + (1-2\nu)\big[n_{x_0}(e_r \Lambda \, \mathrm{d}l)\big]\psi \\
& + (1-2\nu)(e_r n_{x_0}) \, \mathrm{d}l \Lambda \psi + \big[(1-2\nu)(\psi n_{x_0}) + 3(e_r \psi)(e_r n_{x_0})\big] e_r \Lambda \, \mathrm{d}l
\end{aligned}
\tag{22bis}
$$

In the left-hand side of (22) and (22bis), the stress vector $t(x_0, n_{x_0})$ is known for each point $x_0 \in S$. The derivatives $\psi_{i,j}(y)$ appear in the right-hand side, these are the derivatives of the function ψ with respect to three *independent* variables z_1, z_2, z_3, reckoned at a point $y \in S$. In fact, since the restriction of ψ on S is the unknown of the problem i.e. $\psi(y)$ for $y \in S$, one must prove that these $\psi_{i,j}$ are actually reduced to the derivatives of the restriction of ψ on S, with respect to some suitable variables.

Let a parametric representation of S be:

$$
(u, v) \in \Delta \mapsto y \in S = F(u, v) = \left| \begin{array}{l} F_1(u, v) \\ F_2(u, v) \\ F_3(u, v) \end{array} \right. \qquad F \in C^{1,\alpha}(\Delta) \tag{23}
$$

where Δ is a domain of R^2 and the components of F are given in Cartesian coordinates. The parameter u must not be confused with the displacement. For a point z of E_3 close to S, let us consider the coordinate transformation defined by:

$$
z \in E_3 = \left| \begin{array}{l} z_1 \\ z_2 \\ z_3 \end{array} \right. = \left| \begin{array}{l} F_1(u, v) + w.n_1(u, v) \\ F_2(u, v) + w.n_2(u, v) \\ F_3(u, v) + w.n_3(u, v) \end{array} \right. \tag{24}
$$

where $n_y(n_1, n_2, n_3)$ is the normal to S at the point $y \in S$ represented by the couple (u, v). Since $S \in C^{1,\alpha}$, $0 < \alpha \leqslant 1$, every point of S is a regular point, and $F_{,u}$ and $F_{,v}$ are linearly independent, so n_y is well defined:

$$
n_y = \left| \begin{array}{l} n_1(u, v) \\ n_2(u, v) = F_{,u}(u, v) \Lambda F_{,v}(u, v) / \| F_{,u}(u, v) \Lambda F_{,v}(u, v) \| \\ n_3(u, v) \end{array} \right. \tag{25}
$$

The function $\psi(z)$ for $z \in E_3$ is written in the form:

$$
\begin{aligned}
\psi(z) &= \psi(z_1, z_2, z_3) \\
&= \psi(F_1(u, v) + w.n_1(u, v), F_2(u, v) + w.n_2(u, v), F_3(u, v) + w.n_3(u, v)).
\end{aligned}
$$

The restriction of ψ on S is then:

$$
\psi(y \in S) = \psi(F_1(u, v), F_2(u, v), F_3(u, v)) \equiv \phi(u, v) \tag{26}
$$

The unknown of the problem is $\phi(u, v)$; it is to be proved that only the derivatives of $\phi(u, v)$ with respect to either u or v actually appear in the right-hand side of (22) or (22bis). We have:

$$\begin{bmatrix} \phi_{i,u}(u, v) \\ \phi_{i,u}(u, v) \\ \dfrac{\partial \psi_i}{\partial n_z}(y \in S) \end{bmatrix} = \begin{bmatrix} F_{1,u}(u, v) & F_{2,u}(u, v) & F_{3,u}(u, v) \\ F_{1,v}(u, v) & F_{2,v}(u, v) & F_{3,v}(u, v) \\ n_1(u, v) & n_2(u, v) & n_3(u, v) \end{bmatrix} \begin{bmatrix} \psi_{i,1}(y \in S) \\ \psi_{i,2}(y \in S) \\ \psi_{i,3}(y \in S) \end{bmatrix} \tag{27}$$

We recall that: $\psi_{i,j}(y \in S) = \left.\dfrac{\partial \psi_i(z)}{\partial z_j}\right|_{z=y \in S}$

The system (27) can be inverted since $\boldsymbol{F}_{,u}$, $\boldsymbol{F}_{,v}$ and $\boldsymbol{n}$ are linearly independent, the determinant of the system is equal to: $(\boldsymbol{F}_{,u}, \boldsymbol{F}_{,v}, \boldsymbol{n}) = \| \boldsymbol{F}_{,u} \Lambda \boldsymbol{F}_{,v} \|$. We obtain:

$$\begin{bmatrix} \psi_{i,1}(y \in S) \\ \psi_{i,2}(y \in S) \\ \psi_{i,3}(y \in S) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & n_1 \\ A_{21} & A_{22} & n_2 \\ A_{31} & A_{32} & n_3 \end{bmatrix} \begin{bmatrix} \phi_{i,u}(u, v) \\ \phi_{i,v}(u, v) \\ \dfrac{\partial \psi_i}{\partial n_z}(y \in S) \end{bmatrix} \tag{28}$$

Thus:

$$\psi_{i,j}(y \in S) = n_j . \frac{\partial \psi_i}{\partial n_z}(y \in S) + (\ldots) \tag{29}$$

where the parenthesis does not include $\dfrac{\partial \psi_i}{\partial n_z}(y \in S)$, but only $\phi_{i,u}$ and $\phi_{i,v}$. From (29), we have:

$$(\mathrm{grad}_z \psi)(y \in S) \equiv \mathrm{grad}_z \psi(z)\,|_{z=y \in S} = \frac{\partial \psi}{\partial n_z}(y \in S) \otimes \boldsymbol{n}_y + (\ldots) \tag{30a}$$

and:

$$(\mathrm{div}_z \psi)(y \in S) \equiv \mathrm{div}_z \psi(z)\,|_{z=y \in S} = \frac{\partial \psi}{\partial n_z}(y \in S) . \boldsymbol{n}_y + (\ldots) \tag{30b}$$

Using the relations (30a) and (30b), one can easily verify that $\dfrac{\partial \psi}{\partial n_z}(y \in S)$ are actually *not* involved in the kernel of the surface integral of (22) and (22 bis). As was expected, only the derivatives of the restriction of ψ on S, $\phi_{i,u}$ and $\phi_{i,v}$, appear in (22) to (22 bis). This point is important for the effective resolution of the integral equations.

Finally, in the usual case when $\boldsymbol{F}_{,u}$ is perpendicular to $\boldsymbol{F}_{,v}$, one can obtain from (22 bis) the integral equation for a general crack:

$$\begin{aligned} \boldsymbol{t}(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) = {} & \frac{\mu}{8\pi(1-\nu)} pv \int_S \frac{1}{r^2} \cdot \frac{1}{\| \boldsymbol{R}_{,u} \| \cdot \| \boldsymbol{F}_{,v} \|} \cdot \\ & \big\{ 2(\boldsymbol{\Phi}_{,u}, \boldsymbol{e}_r, \boldsymbol{F}_{,v}) \boldsymbol{n}_{x_0} - (1-2\nu)(\boldsymbol{\Phi}_{,u}, \boldsymbol{e}_r, \boldsymbol{n}_{x_0}) \boldsymbol{F}_{,v} - (1-2\nu)(\boldsymbol{F}_{,v} \cdot \boldsymbol{n}_{x_0}) \boldsymbol{\Phi}_{,u} \wedge \boldsymbol{e}_r \\ & \quad + 3(\boldsymbol{e}_r \cdot \boldsymbol{\Phi}_{,u}) \big[(\boldsymbol{F}_{,v}, \boldsymbol{n}_{x_0}, \boldsymbol{e}_r) \boldsymbol{e}_r + (\boldsymbol{n}_{x_0} \cdot \boldsymbol{e}_r) \boldsymbol{e}_r \wedge \boldsymbol{F}_{,v} \big] \\ & \quad - 2(\boldsymbol{\Phi}_{,v}, \boldsymbol{e}_r, \boldsymbol{F}_{,u}) \boldsymbol{n}_{x_0} + (1-2\nu)(\boldsymbol{\Phi}_{,v}, \boldsymbol{e}_r, \boldsymbol{n}_{x_0}) \boldsymbol{F}_{,u} + (1-2\nu)(\boldsymbol{F}_{,u} \cdot \boldsymbol{n}_{x_0}) \boldsymbol{\Phi}_{,v} \wedge \boldsymbol{e}_r \\ & \quad - 3(\boldsymbol{e}_r \cdot \boldsymbol{\Phi}_{,v}) \big[(\boldsymbol{F}_{,u}, \boldsymbol{n}_{x_0}, \boldsymbol{e}_r) \boldsymbol{e}_r + (\boldsymbol{n}_{x_0} \cdot \boldsymbol{e}_r) \boldsymbol{e}_r \wedge \boldsymbol{F}_{,u} \big] \big\} \, \mathrm{d}Sy \\ & \frac{\mu}{8\pi(1-\nu)} \int_L \frac{1}{r^2} \big\{ 2(\boldsymbol{\Psi}, \boldsymbol{e}_r, \boldsymbol{dl}) \boldsymbol{n}_{x_0} + (1-2\nu) \big[(\boldsymbol{e}_r, \boldsymbol{\Psi}, \boldsymbol{n}_{x_0}) \boldsymbol{dl} + (\boldsymbol{dl} \cdot \boldsymbol{n}_{x_0}) \boldsymbol{e}_r \wedge \boldsymbol{\Psi} \big] \\ & \big\{ + 3(\boldsymbol{e}_r \cdot \boldsymbol{\Psi}) \big[(\boldsymbol{n}_{x_0}, \boldsymbol{e}_r, \boldsymbol{dl}) \boldsymbol{e}_r + (\boldsymbol{n}_{x_0} \cdot \boldsymbol{e}_r) \boldsymbol{e}_r \wedge \boldsymbol{dl} \big] \big\}. \end{aligned} \tag{22ter}$$

6. Particular case of a plane crack

Let us consider a crack lying in the plane P (Fig. 3). A natural choice of the coordinate system is so that $\boldsymbol{e}_3$ is the normal orientating P, and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ lie in the plane. We may choose the following parametrization of S:

$$y \in S = \boldsymbol{F}(u, v) = \left| \begin{array}{l} F_1(u, v) \\ F_2(u, v) \\ F_3(u, v) \end{array} \right. = \left| \begin{array}{l} u \\ v \\ 0 \end{array} \right. = \left| \begin{array}{l} y_1 \\ y_2 \\ y_3 = 0 \end{array} \right.$$

which obviously yields:

$$\boldsymbol{F}_{,u} = (1, 0, 0), \ \boldsymbol{F}_{,v} = (0, 1, 0) \quad \text{and } \boldsymbol{n}(u, v) = (0, 0, 1) = \boldsymbol{e}_3$$

From (26), the restriction of ψ on S is:

$$\phi(y_1, y_2) = \psi(y_1, y_2, 0)$$

According to the former discussion, the normal derivative $\dfrac{\partial \psi}{\partial n_z}(y \in S) = \psi_{,3}(y_1, y_2, 0)$ will not be involved, only the derivatives of the *two*-variables function ϕ remain:

$$\phi_{i,1}(y_1, y_2) = \psi_{i,1}(y_1, y_2, 0) \quad \text{and } \phi_{i,2}(y_1, y_2) = \psi_{i,1}(y_1, y_2, 0)$$

As:

$$\boldsymbol{e}_r = (r_{,1}, r_{,2}, 0)$$

$$\boldsymbol{n}_y = \boldsymbol{n}_{x_0} = \boldsymbol{e}_3 = (0, 0, 1)$$

$$\boldsymbol{e}_r \Lambda \mathrm{d}\boldsymbol{l} = \sin\theta . \, \mathrm{d}l . \boldsymbol{n}_{x_0}, \quad \theta \equiv (\boldsymbol{e}_r \hat{;} \, \mathrm{d}l)$$

$$\boldsymbol{n}_{x_0}(\boldsymbol{e}_r \Lambda \, \mathrm{d}\boldsymbol{l}) = \sin\theta . \, \mathrm{d}l$$

(22ter) is reduced to the following in a Cartesian coordinates system:

$$\boldsymbol{t}(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) = \frac{\mu}{8\pi(1-\nu)} pv \int_S \left| \begin{array}{l} \dfrac{1}{r^2}\left[(1-2\nu)(\phi_{1,2}r_{,2} - \phi_{2,2}r_{,1}) + 3r_{,1}r_{,\alpha}\phi_{\alpha,\beta}r_{,\beta}\right] \\ \dfrac{1}{r^2}\left[(1-2\nu)(\phi_{2,1}r_{,1} - \phi_{1,1}r_{,2}) + 3r_{,2}r_{,\alpha}\phi_{\alpha,\beta}r_{,\beta}\right] \\ \dfrac{2}{r^2}\phi_{3,\alpha}r_{,\alpha} . \end{array} \right. \tag{31}$$

$$- \frac{\mu}{8\pi(1-\nu)} \int_L \frac{1}{r^2} \left| \begin{array}{l} \left[(1-2\nu)(\phi \Lambda \boldsymbol{n}_{x_0})\,\mathrm{d}\boldsymbol{l} + 3(\boldsymbol{e}_r \boldsymbol{\phi}) \sin\theta \, \mathrm{d}l\right] r_{,1} + (1-2\nu) \sin\theta \, \mathrm{d}l \phi_1 \\ \left[(1-2\nu)(\phi \Lambda \boldsymbol{n}_{x_0})\,\mathrm{d}\boldsymbol{l} + 3(\boldsymbol{e}_r \boldsymbol{\phi}) \sin\theta \, \mathrm{d}l\right] r_{,2} + (1-2\nu) \sin\theta \, \mathrm{d}l \phi_2 \\ (1+2\nu)(\boldsymbol{\phi} \boldsymbol{n}_{x_0}) \sin\theta \, \mathrm{d}l + (1-2\nu) \sin\theta \, \mathrm{d}l \phi_3 \end{array} \right.$$

where the Greek indices take the value 1 or 2.

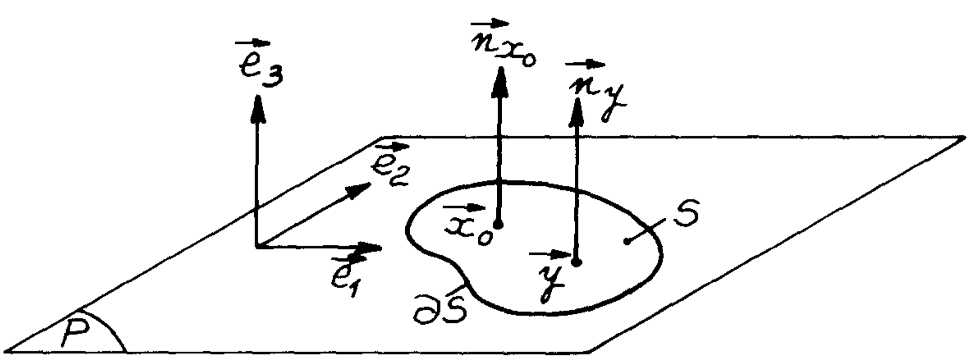

Figure 3. Case of a plane crack.

a) In the case when the crack is embedded in the body, the line integral vanishes, the surface integral taken on S only remains. The third equation of (31) is visibly identical to that of Bui [4]. Let us prove that the first two equations are also equivalent to those of Bui.

We make use of the formulae of integration by parts:

$$\begin{aligned} \iint \psi\phi_{,1}\, \mathrm{d}y_1\, \mathrm{d}y_2 &= -\iint \psi_{,1}\phi\, \mathrm{d}y_1\, \mathrm{d}y_2 + \oint \psi\phi\, \mathrm{d}y_2 \\ \iint \psi\phi_{,2}\, \mathrm{d}y_1\, \mathrm{d}y_2 &= -\iint \psi_{,2}\phi\, \mathrm{d}y_1\, \mathrm{d}y_2 - \oint \psi\phi\, \mathrm{d}y_1 \end{aligned} \tag{32}$$

Having in mind that (22ter) involves a principal value, the surface integral in (32) must be performed over S except a circle $\sigma(\boldsymbol{x}_0, \epsilon)$ with centre at $\boldsymbol{x}_0$, the radius ϵ of the circle tending to zero; the line integral in (32) must be for the same reason taken along the boundary of $\sigma(\boldsymbol{x}_0, \epsilon)$. By these formulae, one may prove for instance:

$$3pv\int_S \frac{r_{,1}r_{,\alpha}r_{,\beta}\phi_{\alpha,\beta}}{r^2}\mathrm{d}S_y = pv\int_S \left(\phi_{1,1}+\phi_{2,2}\right)\frac{r_{,1}}{r^2}\mathrm{d}S_y + pv\int_S \phi_{1,\alpha}\frac{r_{,\alpha}}{r^2}\mathrm{d}S_y \tag{33}$$

One obtains then:

$$\begin{aligned} t_1(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) &= \frac{\mu}{4\pi} pv\int_S \frac{1}{r^2}\phi_{1,\alpha}r_{,\alpha}\, \mathrm{d}S_y + \frac{\mu\lambda}{4\pi(\lambda+2\mu)}\,\cdot \\ & pv\int_S \frac{1}{r^2} r_{,1}\left(\phi_{1,1}+\phi_{2,2}\right)\mathrm{d}S_y \end{aligned} \tag{34}$$

which is identical to that of [4].

It should be noticed that in this case of plane cracks, the mode I is uncoupled from modes II and III.

b) In some special problems, the density could take the form:

$\psi = (0, 0, \psi_3) = \psi_3.\boldsymbol{n}_{x_0}$

The first two equations are identically satisfied, as to the third one, it is reduced to:

$$t_3(\boldsymbol{x}_0, \boldsymbol{n}_{x_0}) = \frac{\mu}{4\pi(1-\nu)} pv\int_S \frac{\phi_{3,\alpha}r_{,\alpha}}{r^2}\mathrm{d}S_y - \frac{\mu}{4\pi(1-\nu)}\int_L \frac{\phi_3 \sin\theta}{r^2}\mathrm{d}l \tag{35}$$

Putot studied in [5] the plane crack at a free surface in opening mode and obtained a system of equations without the line integral: he has considered the crack S made of the union of the surface crack F in question and its mirror image $\bar{F}$ through the plane of the free surface (Fig. 4). The symmetry of the problem, in particular the fact that ψ_3 successively defined on F and $\bar{F}$ takes the same value at each point of L whereas the associated line integrals are taken along opposite directions, implies that the line integral vanishes. When the free surface is not plane, a mirror image is meaningless and the line integral differs from zero.

7. Particular case of a cylindrical crack

Use will be made in this section of cylindrical coordinates (ρ, θ, z). Let us consider a crack curved in the shape of part of a cylindrical surface (Fig. 5), defined by:

$y_1^2 + y_2^2 = R^2 = \text{const.}, \quad \theta \in]-\theta_0, \theta_0[, \; z \in]-z_0, z_0[.$

Following the notations of (23) to (30), we may choose a parametric representation of S:

$$y \in S = \boldsymbol{F}(\theta, z) = \begin{vmatrix} F_1(\theta, z) \\ F_2(\theta, z) \\ F_3(\theta, z) \end{vmatrix} = \begin{vmatrix} R\cos\theta \\ R\sin\theta \\ z \end{vmatrix}$$

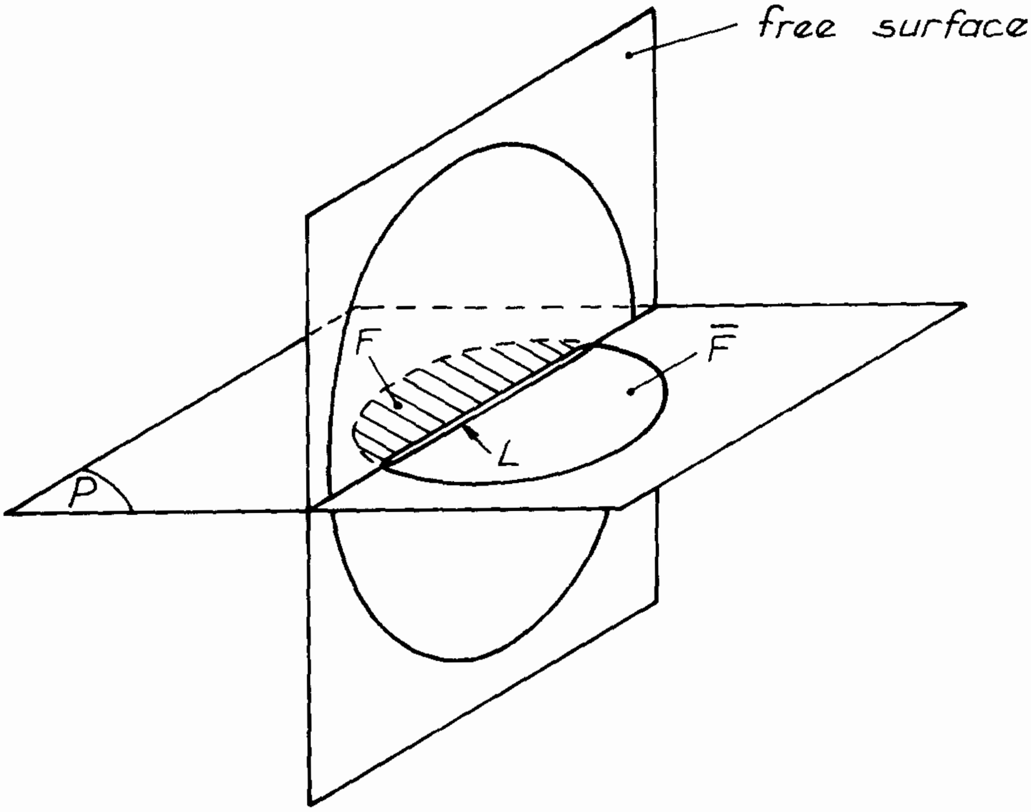

Figure 4. [5] Crossed-cracks configuration

$S = F \cup \bar{F}$

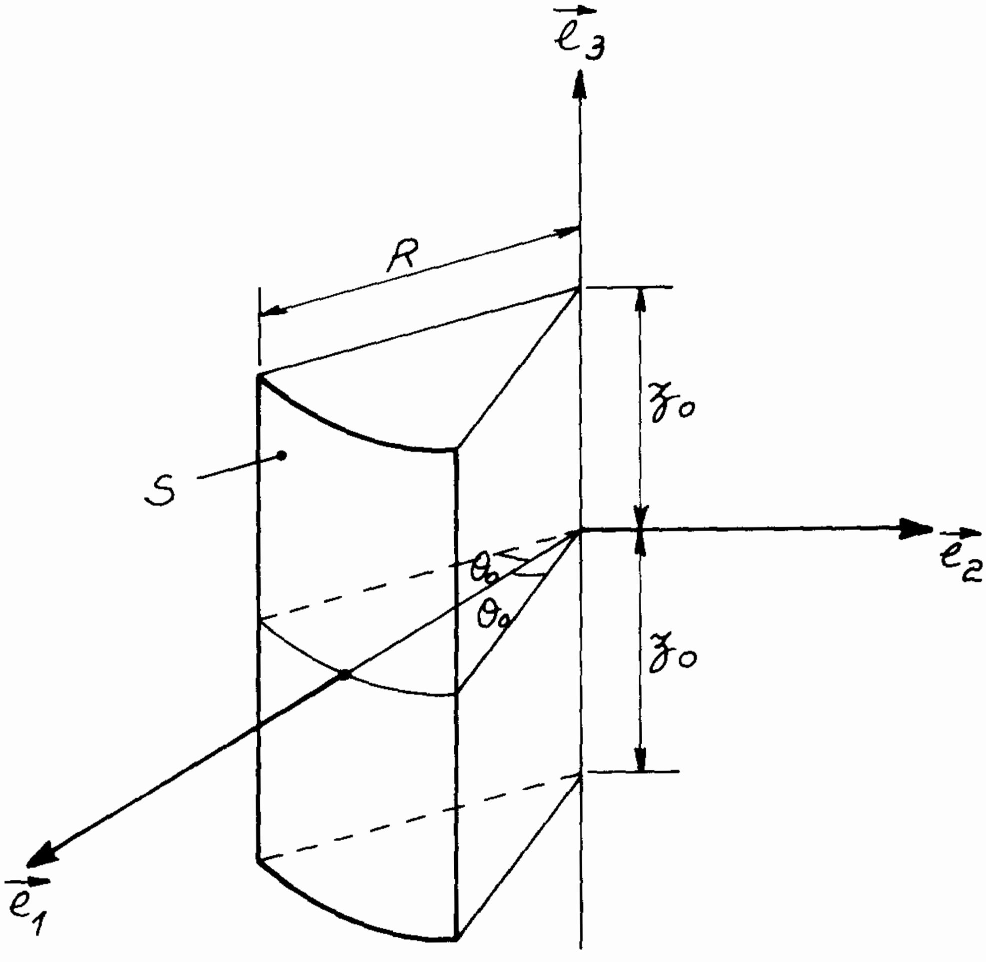

Figure 5. Case of a crack curved in the shape of part of a cylindrical surface.

which yields:

$$F_{,\theta} = Re_\theta,\ F_{,z} = e_3,\ n(\theta,\ z) = n(\theta) = e_\rho \quad \text{and } z \in E_3 = \begin{vmatrix} z_1 \\ z_2 \\ z_3 \end{vmatrix} = \begin{vmatrix} (R+w)\sin\theta \\ (R+w)\sin\theta \\ z \end{vmatrix}$$

as $n = e_\rho$, and $\rho = R + w$, we have:

$$\frac{\partial\psi}{\partial n_z}(y \in S) \equiv \left.\frac{\partial\psi((R+w)\cos\theta,\ (R+w)\sin\theta,\ z)}{\partial w}\right|_{w=0} \equiv \frac{\partial\psi}{\partial\rho}(y \in S)$$

The restriction of ψ on S is:

$$\phi(\theta,\ z) = \psi(R\cos\theta,\ R\sin\theta,\ z)$$

The partial derivatives of ψ with respect to z_1, z_2, z_3, calculated at a point $y \in S$ are expressed by:

$$\begin{bmatrix} \psi_{i,1}(y \in S) \\ \psi_{i,2}(y \in S) \\ \psi_{i,3}(y \in S) \end{bmatrix} = \begin{bmatrix} -\dfrac{\sin\theta}{R} & 0 & \cos\theta \\ \dfrac{\cos\theta}{R} & 0 & \sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{i,\theta}(\theta,\ z) \\ \phi_{i,z}(\theta,\ z) \\ \dfrac{\partial\psi_i}{\partial\rho}(y \in S) \end{bmatrix} \tag{36}$$

It follows from the discussion in §5 that the normal derivative $\psi_{i,\rho}(y \in S)$ must eventually disappear in the integral equations (22) or (22bis), only the derivatives of the restriction ϕ of ψ on S, with respect to θ and z remain.

Equation (36) differs from (29) and (30) of [6], even though it leads to the same final result.

8. Problem of a through crack in a cylindrical thick tube

Let us consider a cylindrical tube with the axis e_3, its outer and inner faces are Σ_1 and Σ_2 respectively, the tube contains a through crack S lying in a cross section (Fig. 6).

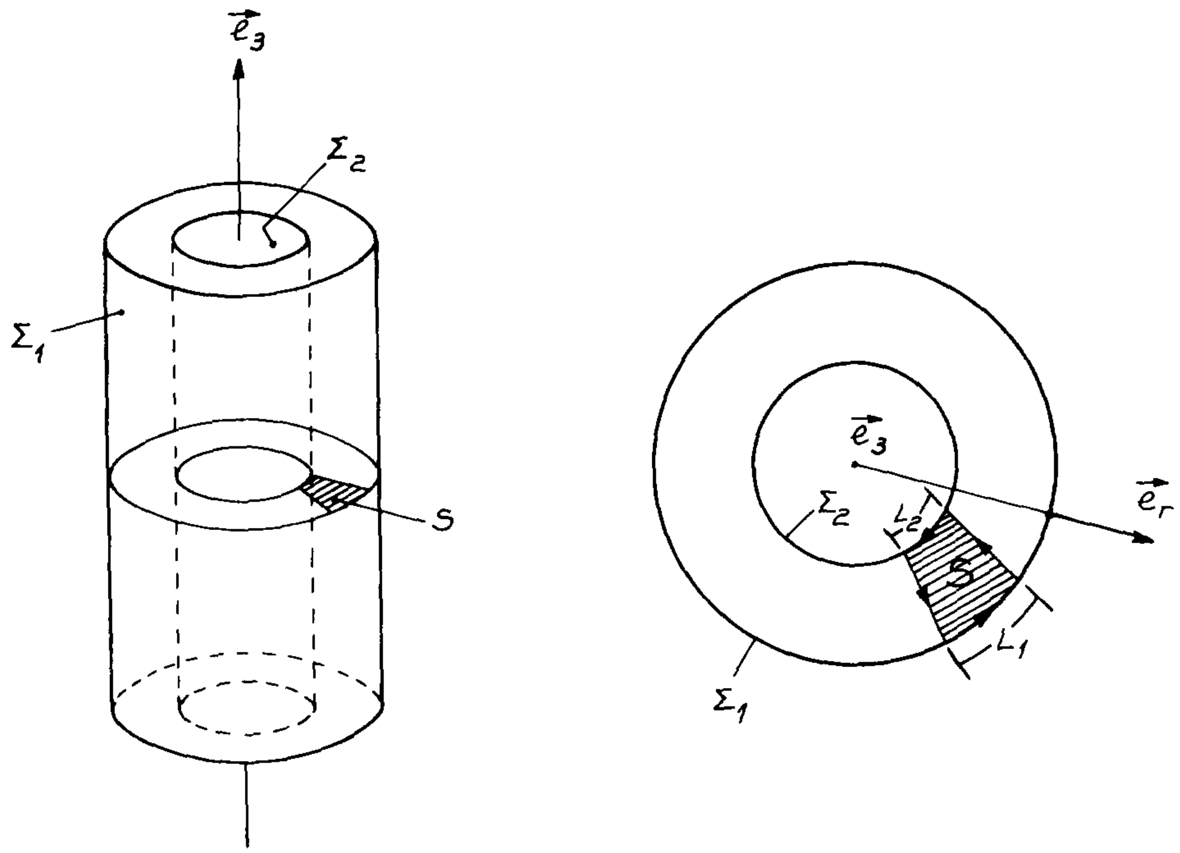

Figure 6. Through crack in a thick tube.

Let:

$$L_1 = \Sigma_1 \cap S$$

$$L_2 = \Sigma_2 \cap S$$

$$L = L_1 \cup L_2$$

It is assumed that the tube is infinite along the e_3-axis, that Σ_1, Σ_2 are free surfaces, and the crack is subjected to the prescribed loading $t(x_0, n_{x_0} = e_3)$.

Let the surface S be oriented by e_3. One can imagine that the two faces Σ_1, Σ_2 are also two crack surfaces, and these three cracks S, Σ_1, Σ_2 are imbedded in an infinite elastic medium. Let us define the displacement field as the sum of three double-layer potentials (cf. (3)):

$$\forall x \in R^3 : u(x) = \int_S \bar{\bar{T}}^T(y - x, n_y)\psi^1(y)\, \mathrm{d}S_y + \int_{\Sigma_1} \bar{\bar{T}}^T(y - x, n_y)\psi^2(y)\, \mathrm{d}\Sigma_y$$

$$+ \int_{\Sigma_2} \bar{\bar{T}}^T(y - x, n_y)\psi^3(y)\, \mathrm{d}\Sigma_y \tag{37}$$

where $n_y = e_3$ for $y \in S$, $n_y = e_\rho$ for $y \in \Sigma_1$, $n_y = -e_\rho$ for $y \in \Sigma_2$.

The densities ψ^1, ψ^2, ψ^3 are defined on three open sets containing respectively S, Σ_1 and Σ_2; their respective restrictions constitute the unknowns of the problems. The boundary conditions are to be specified separately:

a) on the crack surface S: for each point $x_0 \in S$, the prescribed tension $t(x_0, n_{x_0} = e_3)$ must be in equililbrium on the one hand with the stress, (31), yielded by the density ψ^1 on S, on the other hand with the stresses, (19) yielded by the densities ψ^2, ψ^3 on Σ_1 and Σ_2 respectively. The associate integral equation has the form:

$$\forall x_0 \in S \setminus L = S : pv\int_S [\ldots]\, \mathrm{d}S_y - \int_L [\ldots]\, \mathrm{d}l + \mathscr{T}(\partial_x, n_x = e_3)$$

$$\left(\int_{\Sigma_1} \bar{\bar{T}}^T \psi^2\, \mathrm{d}\Sigma_y + \int_{\Sigma_2} \bar{\bar{T}}^T \psi^3\, \mathrm{d}\Sigma_y \right) = t(x_0, n_{x_0} = e_3) \tag{38a}$$

where the brackets [ ], too long to be explicit, stand for the kernel of (31).

b) on the free surface Σ_1: for each point $x_0 \in \Sigma_1$, the sum of the stress generated by ψ^2 on Σ_1,and those by ψ^1 on S and ψ^3 on Σ_2, is equal to zero. This gives the second integral equation which has the form:

$$Vx_0 \in \Sigma_1 \setminus L_1 : pv\int_{\Sigma_1} [\ldots]\, \mathrm{d}\Sigma_y + \mathscr{T}(\partial_x, n_x = e_\rho)$$

$$\left(\int_S \bar{\bar{T}}^T \psi^1\, \mathrm{d}S_y + \int_{\Sigma_2} \bar{\bar{T}}^T \psi^3\, \mathrm{d}\Sigma_y \right) = \mathbf{0} \tag{38b}$$

The first integral given by (22ter) involves now the density ψ^2, we notice that there is no line integral associated with the integral over Σ_1, the free surface being assumed to be infinite along the e_3-axis.

c) on the free surface Σ_2: by the same way we obtain the third integral equations:

$$\forall x_0 \in \Sigma_2 \setminus L_2 : pv\int_{\Sigma_2} [\ldots]\, \mathrm{d}\Sigma_y + \mathscr{T}(\partial_x, n_x = -e_\rho)$$

$$\left(\int_S \bar{\bar{T}}^T \psi^1\, \mathrm{d}S_y + \int_{\Sigma_1} \bar{\bar{T}}^T \psi^2\, \mathrm{d}\Sigma_y \right) = \mathbf{0} \tag{38c}$$

where the first integral involves ψ^3.

Equations (38a) to (38c) constitute the integral equations system of the problem. The coupling between the crack S and the free surfaces is well illustrated.

For numerical purposes, let n_S, n_{Σ_1}, n_{Σ_2} be the respective nodes numbers on S, Σ_1, Σ_2, there are $3\,(n_S + n_{\Sigma_1} + n_{\Sigma_2})$ unknowns which are the values of $\psi_{i/S}$ at the nodes.

At each node, the prescribed stress is given, thus the system (38) becomes an algebraic equation system with $3\,(n_S + n_{\Sigma_1} + n_{\Sigma_2})$ equations.

9. Conclusions

In general cases the derived integral equations system involves both surface and line integrals. Whereas Bui used two Kupradze potentials, a simple-layer and a double-layer, we limited ourselves to only one double-layer potential by analogy with V. and J. Sládek. The latter generates a point singularity of the type r^{-3} that we then transformed into a point singularity of the type r^{-2}, after which we were able to utilize known results of double-layer potentials.

We also demonstrated integral equations are not ill-conditioned, i.e. although the unknown density ψ was formerly defined on an open set containing the crack, only the restriction of ψ and its derivatives with respect to suitable variables appeared in the final integral equations.

By extension of Putot's point of view according to which a free boundary can be considered as an unloaded crack, we dealt with the problem of a finite body containing a three-dimensional arbitrarily shaped crack. It can be a through or a part-through crack.

The example tackled in the last section, devoted to the coupling between a crack and free boundaries, showed that the same reasoning can be applied to more general cases including the particular one of an inclined surface crack.

Numerical applications are being investigated. The relevant main difficulty is the computation of the singular surface integrals taken in the sense of the Cauchy principal value. Another difficulty results from the presence of free boundaries which increase both the size of the equations system and the number of unknowns.

References

[1] I.N. Sneddon, *Proceedings of the Royal Society of London* 187 (1956) 229.

[2] T.A. Cruse, *International Journal of Solids and Structures* 5 (1969) 1259–1274.

[3] V.D. Kupradze, *Progress in Solid Mechanics,* North Holland Pub. (1963).

[4] H.D. Bui, *Comptes rendus de l'Académie des Sciences, Paris,* t. 280, Série A (1975) 1157, and H.D. Bui, *Journal of the Mechanics and Physics of Solids* 25 (1977) 29–39.

[5] C. Putot and H.D. Bui, *Comptes rendus de l'Académie des Sciences, Paris,* t. 288, séie A (1979) and 311. C. Putot, *Advances in Fracture Research,* Pergamon Press, Oxford & New York (1980) 141.

[6] V. Sládek and J. Sládek, *International Journal of Solids and Structures* 19 (1983) 425–436.

[7] V.D. Kupradze, T.G. Gegelia, M.O. Bashelishvili and T.V. Burchuladze, *Three-dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity,* North Holland Pub. (1979).

[8] V. Parton and P. Perline, *Equations Intégrales de la Théorie de l'Élasticité,* Nauka 1977, translated from Russian (1983).

Résumé

A l'aide des potentiels de Kupradzé on formule sous forme d'équations intégrales le problème d'une fissure tridimensionnelle dans un solide fini ou non. Le solide a un comportement élastique linéaire homogène et isotrope et la fissure peut être débouchante ou non ou même traversante. Dans les équations finales apparaissent dans le cas général à la fois des intégrales de surface et des intégrales curvilignes. Pour une fissure débouchante, l'intégrale curviligne est prise sur une portion du front de fissure. Pour une fissure plane, immergée, on retrouve les équations intégrales développées par Bui. Une autre application concerne le cas d'une fissure traversante dans un tube circulaire.